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Asymptotic distribution of the spectra of a class of generalized Kac–Murdock–Szegö matrices

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Abstract

We consider symmetric Toeplitz matrices $T_n = (t_{|r-s|})_{r,s=1}^n$ with $t_r = \alpha\rho^r + \beta/\rho^r$, where α and β are real and $0 < \rho < 1$. We give formulas for $\det(T_n)$ and T_n^{-1}, and show that if $\alpha - \beta = 1$ and $\beta \neq 0$ then T_n has eigenvalues $\lambda_{1n} < \lambda_{2n} < \cdots < \lambda_{nn}$ such that $\lim_{n\to\infty} \lambda_{1n} = -\infty$, $\lim_{n\to\infty} \lambda_{nn} = \infty$, and $\{\lambda_{2n}, \ldots, \lambda_{n-1,n}\}$ are equally distributed as $n \to \infty$ with values of

$$F(\theta) = \frac{1-\rho^2}{1 - 2\rho\cos\theta + \rho^2}$$

at $n-2$ equally spaced points in $[0, \pi]$. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

The Kac–Murdock–Szegö (KMS) matrices [4] are symmetric Toeplitz matrices

$$K_n(\rho) = \left(\rho^{|r-s|}\right)_{r,s=1}^n, \quad n = 1, 2, \ldots,$$

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where $0 < \rho < 1$. Their entries come from the Fourier coefficients of

$$F(\theta) = \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}; \tag{1}$$

specifically, $F(\theta) = \sum_{n=-\infty}^{\infty} \rho^{|n|} e^{in\theta}$. It is known [3, Section 7.2, Problems 12 and 13] that

$$K_n^{-1}(\rho) = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}.$$

From [5, pp. 69 and 70], the eigenvalues of $K_n(\rho)$ are

$$\lambda_{kn} = \frac{1-\rho^2}{1-2\rho\cos\theta_{kn}+\rho^2}, \quad k = 1, 2, \ldots, n,$$

where

$$\frac{(k-1)\pi}{n+1} < \theta_{kn} < \frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n.$$

(See [7] for more on this.) This illustrates a theorem of Szegö [5, ch. 5] (see also [9]) which implies that if $\{c_r\}_{r=-\infty}^{\infty}$ are the Fourier coefficients of a bounded real-valued even function $f \in L[-\pi, \pi]$ then the spectra of the symmetric Toeplitz matrices $T_n = (c_{r-s})_{r,s=1}^{n}$, $n = 1, 2, \ldots,$ are equally distributed in the sense of Weyl [5, p. 62] with values of f at n equally spaced points in $[0, \pi]$, as $n \to \infty$. In fact, a proof like that of Theorem 4 shows that if

$$\psi_{kn} = F\left(\frac{(2k-1)\pi}{2n+2}\right), \quad k = 1, 2, \ldots, n,$$

and G is any function continuous on $[m, M]$, where

$$m = \frac{1-\rho}{1+\rho} = \min_{0 \leqslant \theta \leqslant \pi} F(\theta) \quad \text{and} \quad M = \frac{1+\rho}{1-\rho} = \max_{0 \leqslant \theta \leqslant \pi} F(\theta), \tag{2}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |G(\lambda_{kn}) - G(\psi_{kn})| = 0.$$

This is stronger than the result implied by Szegö's theorem, since $\{\lambda_{kn}\}$ and $\{\psi_{kn}\}$ are equally distributed in Weyl's sense as $n \to \infty$ if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[G(\lambda_{kn})-G(\psi_{kn})]=0$$

(no absolute values) for all G continuous on $[m, M]$.

Szegö's theorem also settles the problem of asymptotic distribution of the spectra of the generalized KMS matrices $T_n=(t_{r-s})_{r,s=1}^{n}$ with $t_r=\sum_{j=1}^{k}c_j\rho_j^r$, where $c_1,\ldots,c_k$ are real and $0<\rho_j<1$, $j=1,\ldots,k$. However, if $\rho_j>1$ for some j then $\sum_{r=1}^{\infty}|t_r|=\infty$, so Szegö's theorem does not apply, and numerical experiments indicate that the asymptotic distribution of the spectra in $\{T_n\}$ is much more complicated.

In this paper we consider the class of generalized KMS matrices

$$T_n=\alpha K_n(\rho)+\beta K_n(1/\rho),\quad n=1,2,\ldots,$$

where α and β are real and $0<\rho<1$; thus, $T_n=(t_{|r-s|})_{r,s=1}^{n}$, where

$$t_r=\alpha\rho^r+\beta\rho^{-r},\quad r=0,\pm1,\pm2,\ldots \tag{3}$$

We give a formula for T_n^{-1} and use it to show that if $\alpha\neq\beta$ and $\beta\neq0$ then the largest and smallest eigenvalues of T_n tend to $\pm\infty$, while the other $n-2$ eigenvalues of T_n are equally distributed as $n\to\infty$ – in the stronger sense mentioned above – with values of $(\alpha-\beta)F$ (see (1)) at $n-2$ equally spaced points in $[0,\pi]$. We believe that this is an interesting – though apparently isolated – example of a family of symmetric Toeplitz matrices whose spectra have predictable "tame" or "Szegö-like" components and "wild" or "un-Szegö-like" components.

2. Determinants and inverses

From (3),

$$t_{r+1}-(\rho+1/\rho)t_r+t_{r-1}=0,\quad -\infty<r<\infty. \tag{4}$$

Let $a=\rho+1/\rho$ and

$$A_n=\begin{bmatrix} a & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & a & -1 & \cdots & 0 & 0 & 0\\ 0 & -1 & a & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a & -1 & 0\\ 0 & 0 & 0 & \cdots & -1 & a & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & a \end{bmatrix}. \tag{5}$$

Since

$$\det(A_n) - (\rho + 1/\rho)\det(A_{n-1}) + \det(A_{n-1}) = 0,$$

$$\det(A_n) = c_1\rho^{n-1} + c_2\rho^{-n+1}, \quad n \geqslant 1 \ (c_1, c_2 \text{ independent of } n).$$

Determining c_1 and c_2 from the initial conditions

$$\det(A_1) = \rho + 1/\rho \quad \text{and} \quad \det(A_2) = (\rho + 1/\rho)^2 - 1$$

yields

$$\det(A_n) = \frac{(1 - \rho^{2n+2})}{\rho^n(1 - \rho^2)}. \tag{6}$$

Theorem 1. *The matrix $T_n = \alpha K_n(\rho) + \beta K_n(1/\rho)$ is nonsingular if and only if*

$$\rho \neq 0, \quad \rho \neq \pm 1, \quad \alpha \neq \beta, \quad \textit{and} \quad \beta^2 - \alpha^2\rho^{2n-2} \neq 0. \tag{7}$$

Proof. We will find $\det(T_n)$. Since

$$K_n(\rho)A_n = \begin{bmatrix} 1/\rho & 0 & 0 & \cdots & 0 & 0 & \rho^n \\ \rho^2 & 1/\rho - \rho & 0 & \cdots & 0 & 0 & \rho^{n-1} \\ \rho^3 & 0 & 1/\rho - \rho & \cdots & 0 & 0 & \rho^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \rho^{n-2} & 0 & 0 & \cdots & 1/\rho - \rho & 0 & \rho^3 \\ \rho^{n-1} & 0 & 0 & \cdots & 0 & 1/\rho - \rho & \rho^2 \\ \rho^n & 0 & 0 & \cdots & 0 & 0 & 1/\rho \end{bmatrix},$$

$$T_nA_n = \begin{bmatrix} t_{-1} & 0 & 0 & \cdots & 0 & 0 & t_n \\ t_2 & t_{-1} - t_1 & 0 & \cdots & 0 & 0 & t_{n-1} \\ t_3 & 0 & t_{-1} - t_1 & \cdots & 0 & 0 & t_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t_{n-2} & 0 & 0 & \cdots & t_{-1} - t_1 & 0 & t_3 \\ t_{n-1} & 0 & 0 & \cdots & 0 & t_{-1} - t_1 & t_2 \\ t_n & 0 & 0 & \cdots & 0 & 0 & t_{-1} \end{bmatrix}. \tag{8}$$

If $t_{-1} \neq 0$ we can add multiples of the first row of $\det(T_nA_n)$ to the other rows to reduce it to upper triangular form. This yields

$$\det(T_nA_n) = (t_{-1} - t_1)^{n-2}\left[t_{-1}^2 - t_n^2\right].$$

Substituting from (3) yields

$$\det(T_nA_n) = -(\alpha-\beta)^{n-2}\rho^{-3n+2}(1-\rho^2)^{n-2}(1-\rho^{2n+2})(\beta^2-\alpha^2\rho^{2n-2}).$$

This and (6) imply that

$$\det(T_n) = -(\alpha-\beta)^{n-2}\rho^{-2n+2}(1-\rho^2)^{n-1}(\beta^2-\alpha^2\rho^{2n-2}).$$

This also holds if $t_{-1} = 0$, since both sides are continuous for all (α, β, ρ). $\square$

Theorem 2. *If* (7) *holds then*

$$T_n^{-1} = \frac{\rho}{(\alpha-\beta)(1-\rho^2)}\begin{bmatrix} a_n & -1 & 0 & \cdots & 0 & 0 & b_n \\ -1 & a & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & a & -1 \\ b_n & 0 & 0 & \cdots & 0 & -1 & a_n \end{bmatrix}, \tag{9}$$

where

$$a_n = \frac{\rho(\beta^2-\alpha^2\rho^{2n-4})}{\beta^2-\alpha^2\rho^{2n-2}} \quad \textit{and} \quad b_n = \frac{\alpha\beta\rho^{n-2}(1-\rho^2)}{\beta^2-\alpha^2\rho^{2n-2}}. \tag{10}$$

Proof. We first note that

$$t_{-1} - t_1 = \frac{(\alpha-\beta)(1-\rho^2)}{\rho} \neq 0. \tag{11}$$

From (8),

$$(T_nA_n)^{-1} = \frac{1}{t_{-1}-t_1}\begin{bmatrix} s_{1n} & 0 & 0 & \cdots & 0 & 0 & s_{nn} \\ s_{2n} & 1 & 0 & \cdots & 0 & 0 & s_{n-1,n} \\ s_{3n} & 0 & 1 & \cdots & 0 & 0 & s_{n-2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{n-2,n} & 0 & 0 & \cdots & 1 & 0 & s_{3n} \\ s_{n-1,n} & 0 & 0 & \cdots & 0 & 1 & s_{2n} \\ s_{nn} & 0 & 0 & \cdots & 0 & 0 & s_{1n} \end{bmatrix} \tag{12}$$

if

$$t_{-1}s_{1n} + t_ns_{nn} = t_{-1} - t_1, \quad t_ns_{1n} + t_{-1}s_{nn} = 0$$

and

$$s_{rn} = \frac{s_{1n}t_r + s_{nn}t_{n-r+1}}{t_1 - t_{-1}}, \quad r = 2, \ldots, n-1.$$

This and (4) imply that

$$s_{r-1,n} - as_{rn} + s_{r+1,n}, \quad r = 3, \ldots, n-2. \tag{13}$$

Since $T_n^{-1} = A_n(T_nA_n)^{-1}$, (5) and (12) imply that

$$\begin{aligned} T_n^{-1} &= A_n(T_nA_n)^{-1} \\ &= \frac{1}{t_{-1} - t_1} \begin{bmatrix} u_{1n} & -1 & 0 & \cdots & 0 & 0 & u_{nn} \\ u_{2n} & a & -1 & \cdots & 0 & 0 & u_{n-1,n} \\ u_{3n} & -1 & a & \cdots & 0 & 0 & u_{n-2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{n-2,n} & 0 & 0 & \cdots & a & 0 & u_{3n} \\ u_{n-1,n} & 0 & 0 & \cdots & -1 & a & u_{2n} \\ u_{nn} & 0 & 0 & \cdots & 0 & -1 & u_{1n} \end{bmatrix}, \end{aligned} \tag{14}$$

where

$$u_{rn} = \frac{s_{r-1,n} - as_{rn} + s_{r+1,n}}{t_1 - t_{-1}}, \quad r = 2, \ldots, n-1.$$

Therefore (13) implies that $u_{rn} = 0$, $r = 3, \ldots, n-2$. Since $\left(T_n^{-1}T_n\right)_{21} = \left(T_n^{-1}T_n\right)_{n-1,1} = 0$,

$$\begin{aligned} t_0u_{2n} + t_{n-1}u_{n-1,n} &= -at_1 + t_2 = -t_0, \\ t_{n-1}u_{2n} + t_0u_{n-1,n} &= -at_{n-2} + t_{n-3} = -t_{n-1}, \end{aligned}$$

so $u_{2n} = -1$ and $u_{n-1,n} = 0$. Since $(T_n^{-1}T_n)_{11} = 1$ and $(T_n^{-1}T_n)_{1n} = 0$,

$$\begin{aligned} t_0u_{1n} + t_{n-1}u_{nn} &= t_{-1} \\ t_{n-1}u_{1n} + t_0u_{nn} &= t_{n-2}, \end{aligned}$$

$$u_{1n} = \frac{t_0t_{-1} - t_{n-2}t_{n-1}}{t_0^2 - t_{n-1}^2} = \frac{\rho(\beta^2 - \alpha^2\rho^{2n-4})}{\beta^2 - \alpha^2\rho^{2n-2}}$$

and

$$u_{nn} = \frac{t_0t_{n-2} - t_{-1}t_{n-1}}{t_0^2 - t_{n-1}^2} = \frac{\alpha\beta\rho^{n-2}(1-\rho^2)}{\beta^2 - \alpha^2\rho^{2n-2}}.$$

This, (11) and (13) imply (9) and (10). □

3. Spectral properties

From [1,2], an $n \times n$ symmetric Toeplitz matrix has $\lceil n/2 \rceil$ symmetric eigenvectors $[x_1 x_2 \ldots x_n]^{\mathrm{T}}$ satisfying $x_{n-r+1} = x_r$, $r = 1, \ldots, n$ and $\lfloor n/2 \rfloor$ skew-symmetric eigenvectors satisfying $x_{n-r+1} = -x_r,\ r = 1, \ldots, n$. We say that an eigenvalue associated with a symmetric (skew-symmetric) eigenvector is *even* (*odd*).

Theorem 3. *If* $\alpha - \beta = 1,\ \beta \neq 0$, *and n is sufficiently large, then the eigenvalues* $\lambda_{1n} < \lambda_{2n} < \cdots < \lambda_{nn}$ *of* T_n *have the following properties*:

(a) *For* $k = 2, 3, \ldots, n-2,\ \lambda_{kn} = F(\theta_{kn})$ *where*

$$\frac{(k-1)\pi}{n-1} < \theta_{kn} < \frac{k\pi}{n-1}. \tag{15}$$

(b) λ_{kn} *alternates between even and odd for* $k = 2,\ 3,\ \ldots,\ n-1$, *with* $\lambda_{n-1,n}$ *even*.

(c) λ_{1n} *is even if* $\beta < 0$, *odd if* $\beta > 0$; *in either case* $\lambda_{1n} < 0$ *and* $\lim_{n\to\infty} \lambda_{1n} = -\infty$.

(d) λ_{nn} *is odd if* $\beta < 0$, *even if* $\beta > 0$; *in either case* $\lim_{n\to\infty} \lambda_{nn} = \infty$.

Proof. If (7) holds then T_n and

$$B_n = \frac{(1-\rho^2)}{\rho} T_n^{-1} = \begin{bmatrix} a_n & -1 & 0 & \cdots & 0 & 0 & b_n \\ -1 & a & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & a & -1 \\ b_n & 0 & 0 & \cdots & 0 & -1 & a_n \end{bmatrix}$$

have the same eigenvectors. Moreover, if γ is an eigenvalue of B_n then

$$\lambda = \frac{1-\rho^2}{\rho\gamma} \tag{16}$$

is an eigenvalue of T_n.

If $x_0,\ x_1,\ \ldots,\ x_{n+1}$ satisfy the difference equation

$$-x_{r-1} + (a-\gamma)x_r - x_{r+1} = 0, \quad 1 \leqslant r \leqslant n, \tag{17}$$

and the boundary conditions

$$(a-a_n)x_1 - b_n x_n = x_0, \quad -b_n x_1 + (a-a_n)x_n = x_{n+1}, \tag{18}$$

then $x = [x_1 x_2 \ldots x_n]^{\mathrm{T}}$ satisfies $B_n x = \gamma x$. Therefore, γ is an eigenvalue of B_n if and only if (17) has a nontrivial solution satisfying (18), in which case x is a γ-eigenvector of B_n.

The solutions of (17) are

$$x_r = c_1\zeta^r + c_2\zeta^{-r}, \tag{19}$$

where c_1 and c_2 are constants and ζ is a zero of the reciprocal polynomial

$$p(z) = z^2 - (a - \gamma)z + 1.$$

Since

$$p(z) = (z - \zeta)(z - 1/\zeta) = z^2 - (\zeta + 1/\zeta)z + 1,$$

if (19) determines an eigenvector of B_n then the corresponding eigenvalue of B_n is

$$\gamma = a - \zeta - \frac{1}{\zeta}. \tag{20}$$

In seeking symmetric eigenvectors of T_n we rewrite (19) as

$$x_r = c\big(\zeta^r + \zeta^{n-r+1}\big). \tag{21}$$

Now $x_n = x_1$ and $x_{n+1} = x_0$, so (18) reduces to $(a - a_n - b_n)x_1 = x_0$, and (21) defines an eigenvector of T_n if and only if $P_n(\zeta) = 0$, with

$$P_n(\zeta) = \sigma_n(\zeta + \zeta^n) - 1 - \zeta^{n+1} \tag{22}$$

and

$$\sigma_n = a - a_n - b_n = \frac{\beta + \alpha\rho^{n+1}}{\rho(\beta + \alpha\rho^{n-1})}. \tag{23}$$

In seeking skew symmetric eigenvectors of T_n we rewrite (19) as

$$x_r = c\big(\zeta^r - \zeta^{n-r+1}\big). \tag{24}$$

Now $x_n = -x_1$ and $x_{n+1} = -x_0$, so (18) reduces to $(a - a_n + b_n)x_1 = x_0$. Therefore (24) defines an eigenvector of T_n if and only if $Q_n(\zeta) = 0$, where

$$Q_n(\zeta) = \tau_n(\zeta - \zeta^n) - 1 + \zeta^{n+1} \tag{25}$$

and

$$\tau_n = a - a_n + b_n = \frac{\beta - \alpha\rho^{n+1}}{\rho(\beta - \alpha\rho^{n-1})}. \tag{26}$$

Since $\beta \neq 0$,

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \tau_n = \frac{1}{\rho} > 1. \tag{27}$$

Now define

$$C_n(\theta) = \sigma_n \cos(n-1)\theta/2 - \cos(n+1)\theta/2, \tag{28}$$

$$S_n(\theta) = \tau_n \sin(n-1)\theta/2 - \sin(n+1)\theta/2. \tag{29}$$

Then $P_n(e^{i\theta}) = 0$ if $C_n(\theta) = 0$ and $Q_n(e^{i\theta}) = 0$ if $S_n(\theta) = 0$. In either case, (16) and (20) imply that $F(\theta)$ is an eigenvalue of T_n.

Because of (27) there is an n_0 such that $\sigma_n > 1$ and $\tau_n > 1$ for $n \geqslant n_0$. Let $\phi_{jn} = j\pi/(n-1)$. Rewriting (28) and (29) as

$$C_n(\theta) = (\sigma_n - \cos\,\theta)\cos(n-1)\theta/2 + \sin\,\theta \sin(n-1)\theta/2,$$
$$S_n(\theta) = (\tau_n - \cos\,\theta)\sin(n-1)\theta/2 - \sin\,\theta\cos(n-1)\theta/2$$

shows that if $n \geqslant n_0$ then C_n changes sign on $(\phi_{2k-1,n}, \phi_{2k,n})$, $k = 1, 2, \dots, \lceil n/2 \rceil - 1$, and S_n changes sign on $(\phi_{2k,n}, \phi_{2k+1,n})$, $k = 1, 2, \dots, \lfloor n/2 \rfloor - 1$. Since F is decreasing, this proves (a) and (b) if (c) and (d) are true for n sufficiently large, which we will now prove.

From (22) and (23), and our assumption that $\alpha - \beta = 1$,

$$P_n(\rho) = \frac{\rho^2 - 1}{\alpha + \beta\rho^{-n+1}},$$

so there is an $n_1 \geqslant n_0$ such that $\beta P_n(\rho) < 0$ for $n \geqslant n_1$. Now suppose $n \geqslant n_1$. Since $P_n(0) = -1$ and $P_n(1) = 2(\sigma_n - 1) > 0$, P_n has a zero ξ_n in $(\rho, 1)$ if $\beta > 0$, or in $(0, \rho)$ if $\beta < 0$. In either case

$$\mu_n = \frac{1 - \rho^2}{1 - \rho(\xi_n + 1/\xi_n) + \rho^2}$$

is an even eigenvalue of T_n, and it is not in $\{F(\theta) \mid 0 \leqslant \theta \leqslant \pi\}$, since $\xi_n + 1/\xi_n > 2$. If $\beta > 0$ then $\mu_n > 0$ (since $\xi > \rho$), so μ_n is the largest eigenvalue of T_n. If $\beta < 0$ then $\mu_n < 0$, so μ_n is the smallest eigenvalue of T_n. Since

$$\xi_n = \frac{1 + \xi_n^{n+1}}{\sigma_n(1 + \xi_n^{n-1})} < \frac{1}{\sigma_n}, \tag{30}$$

(27) implies that $\limsup_{n\to\infty} \xi_n \leqslant \rho < 1$. Now letting $n \to \infty$ in (30) shows that $\lim_{n\to\infty} \xi_n = \rho$. Therefore $\lim_{n\to\infty} |\mu_n| = \infty$.

Now consider

$$R_n(\zeta) = Q_n(\zeta)/(1 - \zeta) = \zeta\tau_n(1 + \zeta + \dots + \zeta^{n-2}) - (1 + \zeta + \dots + \zeta^n).$$

From (25) and (26) and our assumption that $\alpha - \beta = 1$,

$$Q_n(\rho) = \frac{\rho^2 - 1}{\alpha - \beta\rho^{-n+1}},$$

so there is an $n_2 \geqslant n_1$ such that $\beta R_n(\rho) > 0$ for $n \geqslant n_2$. Now choose $n_3 \geqslant n_2$ so that $R_n(1) = (n-1)\tau_n - (n+1) > 0$ for $n \geqslant n_3$. Suppose $n \geqslant n_3$. Since $R_n(0) = -1$, R_n has a zero η_n in $(0, \rho)$ if $\beta > 0$, or in $(\rho, 1)$ if $\beta < 0$. In either case

$$v_n = \frac{1-\rho^2}{1-\rho(\eta_n + 1/\eta_n) + \rho^2}$$

is an odd eigenvalue of T_n, and it is not in $\{F(\theta) \mid 0 \leqslant \theta \leqslant \pi\}$, since $\eta_n + 1/\eta_n > 2$. If $\beta > 0$ then $v_n < 0$ (since $\eta_n < \rho$), so v_n is the smallest eigenvalue of T_n. If $\beta < 0$ then $v_n > 0$ (since $\xi_n < \rho$), so v_n is the largest eigenvalue of T_n. Since

$$\eta_n = \frac{1-\eta_n^{n+1}}{\sigma_n(1-\eta_n^{n-1})} < \frac{n+1}{(n-1)\sigma_n}, \tag{31}$$

(27) implies that $\limsup_{n\to\infty} \eta_n \leqslant \rho < 1$. Now letting $n \to \infty$ in (31) shows that $\lim_{n\to\infty} \eta_n = \rho$. Therefore $\lim_{n\to\infty} |v_n| = \infty$. □

Note that in the limiting case where $\rho = 1$ the situation is quite different, since then $\lambda = 0$ is an eigenvalue with multiplicity $n-1$ and $\lambda_{nn} = n(\alpha+\beta)$ is the only nonzero eigenvalue, assuming that $\alpha + \beta \neq 0$.

Theorem 4. *Let*

$$\chi_{kn} = F\left(\frac{(2k-1)\pi}{2n-2}\right), \quad 2 \leqslant k \leqslant n-1. \tag{32}$$

If G is any continuous function on $[m, M]$ (see (2)) then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n-1} |G(\lambda_{kn}) - G(\chi_{kn})| = 0. \tag{33}$$

Proof. From (15) and (32) and the mean value theorem

$$|\lambda_{kn} - \chi_{kn}| \leqslant \frac{K\pi}{2n-2}, \tag{34}$$

where $K = \max_{0 \leqslant \theta \leqslant \pi} |F'(\theta)|$. Let

$$W_n(G) = \sum_{k=2}^{n-1} |G(\lambda_{kn}) - G(\chi_{kn})|.$$

If G is constant then $W_n(G) = 0$. If N is positive integer then (34) and the mean value theorem imply that

$$|\lambda_{kn}^N - \chi_{kn}^N| \leqslant NM^{N-1} |\lambda_{kn} - \chi_{kn}| \leqslant \frac{NM^{N-1}K\pi}{2n-2},$$

so (33) holds if G is a polynomial.

Now suppose G is an arbitrary continuous function on $[m, M]$ and let $\epsilon > 0$ be given. From the Weierstrass approximation theorem, there is a polynomial P such that $|G(u) - P(u)| < \epsilon$ for all u in $[\alpha, \beta]$. Therefore $W_n(G) < W_n(P) + 2n\epsilon$, and

$$\limsup_{n\to\infty} \frac{W_n(G)}{n} \leqslant \lim_{n\to\infty} \frac{W_n(P)}{n} + 2\epsilon = 2\epsilon.$$

Now let $\epsilon \to 0$ to conclude that $\lim_{n\to\infty} W_n(G)/n = 0$. $\square$

4. The case where $\alpha = \beta$

For completeness we now consider the case where $\alpha = \beta = 1$. From (8) and (11) we see that $\lambda = 0$ is an eigenvalue of multiplicity $n - 2$. Since

$$t_r = \rho^{|r-s|} + \rho^{-|r-s|} = \rho^{r-s} + \rho^{s-r}, \tag{35}$$

the vectors

$$\begin{bmatrix} 1 \\ -a \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -a \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -a \\ 1 \end{bmatrix}$$

form a basis for the null space of T_n. We will show that the vectors $u = [u_1\, u_2 \cdots u_n]^{\mathrm{T}}$ and $v = [v_1\, v_2 \cdots v_n]^{\mathrm{T}}$ with

$$u_r = \rho^{r-(n+1)/2} + \rho^{-r+(n+1)/2} \quad \text{and} \quad v_r = \rho^{r-(n+1)/2} - \rho^{-r+(n+1)}/2,$$

are eigenvectors associated with the nonzero eigenvalues, which we will compute.

It is straightforward to verify that

$$t_{r-s} = \frac{u_r u_s - v_r v_s}{2}$$

(see (35)) and $(u, v) = 0$. Therefore

$$\sum_{s=1}^{n} t_{r-s} u_s = \frac{\|u\|^2 u_r}{2} \text{ and } \sum_{s=1}^{n} t_{r-s} v_s = -\frac{\|v\|^2 v_r}{2}, \quad r = 1, \ldots, n,$$

so u is an eigenvector associated with

$$\lambda_{nn} = \frac{\|u\|^2}{2} = n + \frac{\rho^{-n+1} - \rho^{n+1}}{1 - \rho^2}$$

and v is an eigenvector associated with

$$\lambda_{1n} = -\frac{\|v\|^2}{2} = n - \frac{\rho^{-n+1} - \rho^{n+1}}{1 - \rho^2}.$$

Therefore $\lim_{n\to\infty} \lambda_{nn} = \infty$ and $\lim_{n\to\infty} \lambda_{1n} = -\infty$. Since u is symmetric and v is skew symmetric, λ_{nn} is even and λ_{1n} is odd.

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